MATH20132 Calculus of Several Variable. 2022-23

Problems 11 Extrema & Saddle Points

1. Suppose

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

is a matrix with real entries. Prove

i. If $\det M > 0$ (in particular $a \neq 0$) then

  a. M is positive definite if $a > 0$;

  b. M is negative definite if $a < 0$.

ii. If $\det M < 0$, then M is indefinite.

iii. If $\det M = 0$ then M is nondefinite.

2. Find the critical points of the following functions.

i. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(\mathbf{x}) = x^3 + x - 4xy - 2y^2$;

ii. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(\mathbf{x}) = x(y+1) - x^2y$;

iii. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(\mathbf{x}) = x^3 - 6xy + y^3$;

iv. $f : \mathbb{R}^3 \to \mathbb{R}$, $f(\mathbf{x}) = x^4 + z^4 - 2x^2 + y^2 - 2z^2$;

v. $f : \mathbb{R}^3 \to \mathbb{R}$, $f(\mathbf{x}) = x^2 + y^2 + z^2 + 2xyz$.

Use the Hessian matrix to determine whether each critical point is a local maximum, a local minimum or a saddle point.